ISRAEL JOURNAL OF MATHEMATICS **122** (2001), 125–147

ON THE FAILURE OF THE CO-HOPF PROPERTY FOR SUBGROUPS OF WORD-HYPERBOLIC GROUPS

BY

ILYA KAPOVICH

Department of Mathematics, University of Illinois
Urbana, IL 61801, USA
e-mail: kapovich@math.uiuc.edu

AND

DANIEL T. WISE*

Department of Mathematics, Malott Hall
Cornell University, Ithaca, NY 14853, USA
e-mail: daniwise@math.cornell.edu

ABSTRACT

We provide an example of a finitely generated subgroup H of a torsion-free word-hyperbolic group G such that H is one-ended, and H does not split over a cyclic group, and H is isomorphic to one of its proper subgroups.

1. Introduction

A group H is said to be **co-Hopfian** provided that any injective endomorphism $H \to H$ is an automorphism. Many familiar groups are not co-Hopfian. For instance, any finitely generated infinite abelian group is not co-Hopfian. Any free product of two nontrivial groups is not co-Hopfian.

Finite groups are obviously co-Hopfian, but there are more interesting examples: If M is a closed aspherical n-dimensional manifold with $\chi(M) \neq 0$, then $\pi_1 M$ is co-Hopfian. Indeed, if ρ: $\hat{M} \to M$ is a cover of M with $\pi_1 \hat{M} \cong \pi_1 M$, then

* Supported as an NSF Postdoctoral Fellow under grant no. DMS-9627506.
Received July 4, 1999

$\hat{M}$ is aspherical because M is aspherical, and so $\hat{M}$ and M are homotopy equivalent. But then $\hat{M}$ must be a finite cover because $\mathsf{H}_n(\hat{M}, \mathbb{Z}_2) = \mathsf{H}_n(M, \mathbb{Z}_2) \neq 0$. And ρ must be a degree 1 cover because, $\chi(\hat{M}) = \text{degree}(\rho)\chi(M)$. One can show similarly that if M is a finite volume hyperbolic 3-manifold, then $\pi_1 M$ is co-Hopfian. There, Mostow rigidity makes the hypothesis that $\chi(M) \neq 0$ unnecessary.

K. Ohshika and L. Potyagailo [OP98] studied the co-Hopf property and examples of its failure for geometrically finite groups of isometries of $\mathbb{H}^n$ (see also [Pot97]). Recently T. Delzant and L. Potyagailo [DP98] proved that a torsion-free geometrically finite non-elementary group G of isometries of $\mathbb{H}^n$ is co-Hopfian if and only if G does not split over an elementary subgroup (possibly trivial) which has infinite index in a maximal elementary subgroup of G.

One of the fundamental results regarding co-Hopficity is the following beautiful theorem of Z. Sela [Sel97]:

THEOREM 1.1: *Let G be a torsion-free freely indecomposable non-elementary word-hyperbolic group. Then G is co-Hopfian.*

(A subgroup Γ of a word-hyperbolic group is called **elementary** if Γ has a cyclic subgroup of finite index. Otherwise Γ is called **non-elementary**.) The proof of this theorem is very complicated and relies heavily on the theory of JSJ decomposition for word-hyperbolic groups developed by Z. Sela in [Sel97].

However, in an earlier paper [RS94] E. Rips and Z. Sela prove a simpler theorem. Namely, they show that if G is a non-elementary torsion-free word-hyperbolic group which admits no nontrivial cyclic splittings, then G is co-Hopfian. (In fact E. Rips and Z. Sela proved this statement for a word-hyperbolic group with no small action on an $\mathbb{R}$-tree, which by a theorem of E. Rips [BF95] for a torsion-free word-hyperbolic group is equivalent to having no cyclic splittings.) Recall that a **cyclic splitting** of G is a decomposition of G as the fundamental group of a graph of groups with cyclic edge groups.

In this note we show that the above statements do not hold for finitely generated subgroups of torsion-free word-hyperbolic groups. More precisely, we show that there is a finitely generated non-elementary freely indecomposable subgroup H of a torsion-free word-hyperbolic group, such that H has no nontrivial cyclic splittings and such that H is not co-Hopfian.

If we drop the requirement that H have no cyclic splittings and just insist that H be freely indecomposable and not co-Hopfian, then Rips's construction [Rip82] can be used to produce such examples. Given a finitely presented group Q, Rips' construction gives a short exact sequence $1 \to N \to G \to Q \to 1$ such that

$N = \langle x, y \rangle$ is two-generated and such that G is a torsion-free finitely presented $C'(\frac{1}{6})$ group and therefore word-hyperbolic. Apply Rips' construction to the Baumslag–Solitar group $Q = \langle a, t \mid t^{-1}at = a^2 \rangle$, and let $\hat{a}$ and $\hat{t}$ be elements of G which map to a and t. Then the subgroup $H = \langle N, \hat{a} \rangle$ is the preimage of $\langle a \rangle$ and it is conjugated by $\hat{t}$ to $\langle N, \hat{a}^2 \rangle$. The subgroup $\langle N, \hat{a}^2 \rangle$ has index 2 in H because it is the preimage of $\langle a^2 \rangle$ which is of index two in $\langle a \rangle$. Thus H is not co-Hopfian.

Observe now that N is non-elementary. Indeed, $N \neq 1$ since $N = 1$ implies $G \cong Q$, which is impossible because the Baumslag–Solitar group Q is not word-hyperbolic. Also, if N is virtually infinite cyclic then N is infinite cyclic itself (since G is a torsion-free word-hyperbolic group). Since N is normal in G and is infinite cyclic, this implies that N has finite index in G [Aea91], contradicting the infiniteness of Q. Thus N is non-elementary and therefore so is H.

Finally, it is easy to see that H is freely indecomposable since H is torsion-free and possesses a proper finitely generated normal subgroup of infinite index, namely N [Bau66].

It turns out that one can provide such an example where H is not just freely indecomposable but also does not split over cyclic groups. More precisely, our main result is the following:

THEOREM 1.2: *There exist a torsion-free word-hyperbolic group G and a subgroup H of G such that the following holds.*

(i) *The group H is two-generated and non-elementary.*

(ii) *The group H does not admit a nontrivial cyclic splitting (and so in particular H is freely indecomposable and one-ended).*

(iii) *The group H is conjugate in G to a proper subgroup of itself. In particular H is not co-Hopfian.*

We now describe our plan for constructing such H. We begin by giving (in Section 3) a refined analysis of a construction used in [Wis98]. Consider the presentation

$$\langle a_1, \ldots, a_r, t \mid w_1, \ldots, w_s, a_1^t = v_1, \ldots, a_r^t = v_r \rangle$$

where w_j and v_i are freely and cyclically reduced words in $a_i^{\pm 1}$. And suppose that the presentation satisfies the $C'(\frac{1}{5})$ small-cancellation condition. In [Wis98] it was shown that the subgroup $H = \langle a_1, \ldots, a_r \rangle$ is not finitely presentable. In Theorem 3.1 we actually produce an explicit infinite presentation for H.

Next, we show in Theorem 4.3 that if the presentation of G satisfies some additional assumptions then the infinite presentation obtained for H satisfies the $C(6)$ small-cancellation condition. Note that any subgroup of a small-cancellation group is the fundamental group of a small-cancellation complex, because we can take the corresponding based cover. The strength and utility of this theorem derives from the fact that we obtain a small-cancellation presentation relative to the original finite set of generators.

(We believe that Theorem 3.1 and Theorem 4.3 are results of independent interest with some potential future applications.)

This $C(6)$ presentation is then used in Theorem 5.1 to see that the injective endomorphism of H which is induced by conjugation by t in G is not a surjection.

Then in Theorem 5.3 we apply the theorems to the specific example

$$\left\langle a, b, t \;\middle|\; \begin{array}{ll} a = [a,b][a^2,b^2]\cdots[a^{100},b^{100}], & a^t = abab^2\cdots ab^{100}a, \\ b = [b,a][b^2,a^2]\cdots[b^{100},a^{100}], & b^t = baba^2\cdots ba^{100}b \end{array} \right\rangle.$$

We use the fact that $\mathrm{rank}(H) = 2$ and that H is perfect to show that H does not split over a cyclic group.

Finally, in Section 6 we outline an alternative approach for constructing an example of a non-co-Hopfian subgroup H of a word-hyperbolic group such that H has no essential cyclic splittings. (A cyclic splitting is called **essential** if every edge group has infinite index in both the vertex groups corresponding to the endpoints of this edge.) The method utilizes the fact that the free Burnside group on two generators of exponent 667 is known to be infinite and not co-Hopfian [Adi79].

Such a construction is of interest since, as it is observed in [Sel97], the original argument of [RS94] can be modified to show that if G is a non-elementary torsion-free word-hyperbolic group which admits no nontrivial essential cyclic splittings, then G is co-Hopfian.

All of the non-co-Hopfian subgroups of word-hyperbolic groups which we construct in this paper are finitely generated but not finitely presentable. This is not surprising since the ambient word-hyperbolic groups in our examples are small-cancellation groups and therefore, by a theorem of S. Gersten [Ger96], all their finitely presentable subgroups are themselves word-hyperbolic and thus covered by the result of Z. Sela.

However, N. Brady [Bra] showed that there exist finitely presentable non-hyperbolic subgroups of word-hyperbolic groups. Thus the following question is still open and appears to be very difficult:

PROBLEM 1.3: *Let H be a torsion-free one-ended finitely presentable subgroup of a word-hyperbolic group. Is H co-Hopfian?*

A group H is said to be Hopfian if every surjective endomorphism of H is an automorphism. Z. Sela proved in [Sel99] that every word-hyperbolic group is Hopfian. It seems reasonable to pose the following problem:

PROBLEM 1.4: *Let H be a finitely generated subgroup of a word-hyperbolic group. Is H Hopfian?*

It seems that it would be difficult to produce a non-Hopfian example. Indeed, if H were not Hopfian, then H and thus G would not be residually finite because, as proved by Malćev, every finitely generated residually finite group is Hopfian [LS77]. It is currently unknown whether there is a word-hyperbolic group which is not residually finite.

2. Small-cancellation groups

We now review some of the basic definitions of small-cancellation theory. A more detailed account is given in [LS77].

Let $F = F(A)$ be a free group on $A = \{a_1, \ldots, a_r\}$. For every $f \in F$ we denote by $|f|$ the length of the freely reduced word in $A^{\pm 1}$ representing f.

If w is a freely and cyclically reduced word in $F(A)$, we define the **cyclic word** $[w]$ to be the set of all cyclic permutations of w. A freely reduced word v in $F(A)$ is said to be a subword of the cyclic word $[w]$ if v is a subword of a cyclic permutation of w.

Definition 2.1 (Small-cancellation): Let P be a set of nontrivial cyclic words in $A^{\pm 1}$ closed under taking inverses.

We say that a freely reduced word v is a **piece** with respect to P if at least one of the following holds:

1. v is a subword of $[w]$ and of $[z]$ where $[w] \neq [z]$ and $[w], [z] \in P$;
2. for some cyclically reduced words vu', vu'' with $u' \neq u''$ we have that $[vu'] = [vu'']$ and $[vu'] \in P$.

Let $\alpha > 0$. We say that P satisfies the $C'(\alpha)$ **small-cancellation condition** if whenever a piece v is a subword of $[w] \in P$, then $|v| < \alpha |w|$.

Let $n > 1$ be an integer. We say that P satisfies the $C(n)$ **small-cancellation condition** if no $w \in P$ can be represented as a concatenation of fewer than n pieces. Note that $C'(1/n) \Rightarrow C(n+1)$.

Also, we say that a group presentation $G = \langle A \,|\, R \rangle$ (where R is a set of reduced, cyclically reduced words in $A^{\pm 1}$) satisfies the $C'(\alpha)$ (respectively $C(n)$) condition if the set $\{[r^{\pm 1}], r \in R\}$ satisfies the $C'(\alpha)$ (respectively $C(n)$) condition. Similarly, we will say that a word v is a piece with respect to the presentation $G = \langle A \,|\, R \rangle$ if v is a piece with respect to the set $\{[r^{\pm 1}], r \in R\}$.

By the **standard 2-complex** of a presentation $\langle A|\ R\rangle$ we mean a 2-complex X such that $X^{(1)}$ is a bouquet of circles on A, and the 2-cells of X are attached along paths corresponding to the relators in R. Since $\pi_1 X \cong \langle A\, R\rangle$, we will move freely between a presentation and its standard 2-complex X. Similarly, we will move freely between words in $A^{\pm 1}$ and their corresponding combinatorial paths in X.

Let W be a word in the generators $a_i^{\pm 1}$ of a presentation, and suppose that W represents the identity element. Then W corresponds to a path $P \to X$ where X is the standard 2-complex of the presentation. A **disc-diagram** for P (or W) is a planar, simply connected 2-complex D and a map $D \to X$ such that the closed path $P \to X$ factors through a closed path $P \to \partial D \subset D$. The map $P \to \partial D$ is a surjection, and furthermore for each open 1-cell e of ∂D, the preimage of e in P consists of one or two open 1-cells according as to whether or not e lies on the boundary of a 2-cell of P. It is a theorem of van Kampen that such disc-diagrams always exist provided that $P \to X$ is null-homotopic (see [LS77]).

If the words U and V represent the same element of $\langle A\, R\rangle$, then by a disc-diagram for $U = V$ we will mean a disc-diagram for the word UV^{-1}.

A disc-diagram $D \to X$ is said to be **reduced** provided that the following condition holds: If C_1 and C_2 are 2-cells of D which meet along a 1-cell e of D, then the boundary cycles of C_1 and C_2 beginning with e (in the same direction) are sent to distinct paths in X after composing with $D \to X$. It is a fact that a reduced diagram exists for any word representing the identity in the presentation (see [LS77]).

A **spur** in a disc-diagram is a 1-cell which ends at a 0-cell of valence 1.

We now recall a version of Greendlinger's Lemma which is the main tool of small-cancellation theory. This sort of theorem was first proven in [Gre60]. A version appears in [LS77] in the 'metric' $C'(\frac{1}{6})$ case. The following 'non-metric' version can be proven similarly (for more details see, for instance, [MW]).

THEOREM 2.2 (Greendlinger's Lemma): *Let X be the standard 2-complex of a $C(6)$ presentation. Let $D \to X$ be a reduced disc-diagram. Suppose that D has no spurs. Then either D consists of a single 0-cell, or D consists of a single 2-cell, or there exist at least two 2-cells C_1, C_2 in D with the following property: for each i, the boundary cycle ∂C_i is the concatenation of two paths $Q_i S_i$ such that S_i is a subpath of the boundary cycle of D and Q_i is the concatenation of at most 3 pieces in D.*

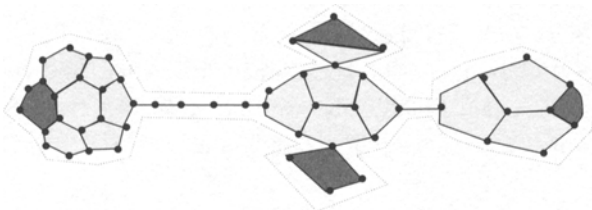

Figure 1. Illustrated above is a singular disc-diagram. The dotted path around the disc-diagram is meant to suggest the route of the boundary cycle of the disc-diagram. The shaded 2-cells in the diagram are complements of ≤ 3 pieces. At least two such 2-cells are guaranteed by Greendlinger's Lemma.

3. Presentation for the subgroup H

We use the notation $x^y = y^{-1}xy$. We let $|P|$ denote the length of a word or combinatorial path P.

THEOREM 3.1: *Let G be presented by*

$$\langle a_1, \ldots, a_r, t \mid w_j, a_i^t = \phi(a_i) : 1 \leq i \leq r, 1 \leq j \leq s\rangle$$

where w_j are freely and cyclically reduced words in the $a_i^{\pm 1}$, and where ϕ is some endomorphism of the free group on a_i. Suppose that the presentation for G satisfies the $C'(\frac{1}{5})$ small-cancellation condition. Let H be the subgroup generated by the a_i. Then H can be presented by $\langle a_i \mid w_{j,n} : n \geq 0, 1 \leq j \leq s\rangle$ where, for $n \geq 0$, we define $w_{j,n}$ to be $\phi^n(w_j)$.

Proof: Let X denote the standard 2-complex of G. Consider a reduced disc-diagram D in X such that ∂D consists exclusively of a_i edges.

Observe that since t does not appear on ∂D, the relators of the form $a_i^t = \phi(a_i)$ form **rings** in D.

We say that a ring points **outwards** or **inwards** depending on whether its t edges are oriented towards or away from ∂D. See Figure 2 for an illustration of some rings in D.

The fundamental point is that all rings point outwards. To see this, consider a reduced disc-diagram D whose exterior 2-cells (meaning those 2-cells whose boundaries contain a 1-cell of ∂D) form a ring R which points inwards. Greendlinger's Lemma provides a 2-cell C in D such that the boundary cycle

∂C of C is the concatenation of two paths UV, where U is a subpath of the boundary path of D and V is the concatenation of at most 3 pieces in X. But each exterior 2-cell of D intersects ∂D in a single a_i edge, and consequently U consists of a single a_i edge. Finally, observe that since V is the concatenation of at most 3 pieces of ∂C, we have $|V| < \frac{3}{5}|\partial C|$, and so $1 = |U| > \frac{2}{5}|\partial C|$ which is impossible.

We now prove the theorem by showing that any freely and cyclically reduced word P in the a_i which is trivial in $\pi_1 X$ is actually freely equivalent to a product of conjugates of $w_{j,n}^{\pm 1}$, where the conjugators are words in the $a_i^{\pm 1}$.

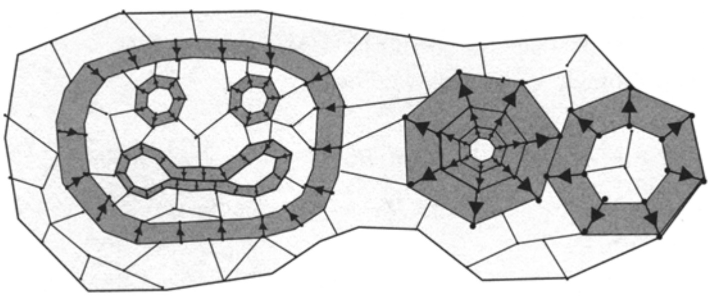

Figure 2. Illustrated above is a disc-diagram in a presentation $\langle a_i, t \mid w_j, a_i^t = v_i \rangle$ where w_j and v_i are words isn the a_i (there is no small-cancellation hypothesis on the presentation). The boundary cycle of the disc-diagram is a word in $a_i^{\pm}$ and therefore the $a_i^t = v_i$ relators form the shaded rings. All of the rings point outwards except for outermost ring of the face, and the mouth.

The **depth** of a disc-diagram is the maximum number of concentric rings. We prove the lemma by induction on the minimal depth of a disc-diagram for P. The lemma is obviously true when the depth of D is 0. Suppose that the statement is true when P has a disc-diagram of depth $< d$ and consider a disc-diagram D whose depth is d. Observe that P is the product of conjugates of outer-paths around outermost rings of D and around 2-cells corresponding to $w_j^{\pm 1}$, and observe that the conjugators are elements of the free group on a_i.

Now, any outermost ring R of D has the property that the inner boundary of R bounds a disc diagram D' whose depth is $< d$. By the inductive hypothesis, the boundary path P' of D' is freely equivalent to the product of conjugates of words $w_{j,n}^{\pm 1}$ where the conjugators are words in the $a_i^{\pm 1}$. The lemma now follows because the outer-path of R is the word $\phi(P')$ which is freely equivalent to the product of conjugates of words $w_{j,n}^{\pm 1}$, where the conjugators are again elements of the free group on a_i. This is because

$$\phi\Big((w_{j_1,n_1}^{\pm 1})^{\alpha_1}\cdots(w_{j_k,n_k}^{\pm 1})^{\alpha_k}\Big) = (w_{j_1,(n_1+1)}^{\pm 1})^{\phi(\alpha_1)}\cdots(w_{j_k,(n_k+1)}^{\pm 1})^{\phi(\alpha_k)}. \qquad \blacksquare$$

Remark 3.2: The proof of Theorem 3.1 actually works under the assumptions that each a_i is a piece, and that the presentation satisfies the non-metric $C(6)$ small-cancellation condition

Remark 3.3: It was shown in [Wis98] that if the standard 2-complex of the presentation for G is aspherical, then the subgroup H is not finitely presentable and hence G is incoherent. This is the case when the presentation for G is $C(6)$ provided that: no relator is conjugate to another relator or to the inverse of another relator, and none of the relators are proper powers [LS77]. In the next section, under some additional hypotheses on G we prove the stronger result that the presentation for H is small-cancellation with infinitely many relators (but finitely many generators).

We conclude by giving examples which show that Theorem 3.1 can fail if we drop the small-cancellation hypothesis.

Example 3.4: Let $G = \langle a, b, t \mid [aa, bb], a^t = aa, b^t = bb\rangle$. It is easy to see that G can also be presented as $\langle a, b, t \mid [a, b], a^t = aa, b^t = bb\rangle$. Then G splits as an HNN extension [LS77], and one sees immediately that H is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. However, the presentation for H which would be provided by Theorem 3.1 is the following $\langle a, b \mid [a^{2n}, b^{2n}] : n \geq 1\rangle$ which is the group $\langle a, b \mid [a^2, b^2]\rangle$; this one can easily show is not isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

More generally, we can choose the presentation $\langle a_1, \ldots, a_r \mid a_i^t = \phi(a_i), \phi(w)\rangle$ where w is a nontrivial element of the free group on a_i. When we apply Theorem 3.1 to this presentation, the derived presentation $\langle a_1, \ldots, a_r \mid \phi^n(w) : n \geq 1\rangle$ is often an incorrect presentation for H because the word w which is trivial in G may not be trivial in the derived presentation.

In particular, this can happen when $G = \langle a_1, \ldots, a_r \mid a_i^t = \phi(a_i), \phi(w)\rangle$ has a more efficient small-cancellation presentation $G = \langle a_1, \ldots, a_r \mid a_i^t = \phi(a_i), w\rangle$ that is small-cancellation.

4. Small-cancellation of the presentation for H

Definition 4.1 (Immersion): We say that an endomorphism ϕ: $F(A) \longrightarrow F(A)$ is an **immersion** if for each $a_i \in A$ $\phi(a_i) \neq_F 1$ and for every $x, y \in A \cup A^{-1}$

such that $x \neq y^{-1}$, we have $|\phi(x)\phi(y)| = |\phi(x)| + |\phi(y)|$ (that is, the product $\phi(x) \cdot \phi(y)$ is freely reduced).

Immersions are a rich class of endomorphisms of $F(A)$. For example, if $\phi\colon F(A) \longrightarrow F(A)$ has the property that for every i the word $\phi(a_i)$ begins and ends with the letter a_i, then ϕ is easily seen to be an immersion. Note that $\phi\colon F(A) \longrightarrow F(A)$ is an immersion if and only if the induced map $B \to B$ is an immersion (meaning local-injection), where B is the bouquet of circles on A.

Definition 4.2 (Condition D): Let $v_1, \ldots, v_r, w_1, \ldots, w_s$ be nontrivial, freely and cyclically reduced words in $A^{\pm 1}$, where $s \geq 1$. We say that the sequence $v_1, \ldots, v_r, w_1, \ldots, w_s$ satisfies condition D if the following holds:

1. The endomorphism $\phi\colon F(A) \longrightarrow F(A)$ defined as $\phi(a_i) = v_i$ is an immersion.
2. The infinite cyclic subgroups of $F(A)$
$$\langle v_1 \rangle, \ldots, \langle v_r \rangle, \langle w_1 \rangle, \ldots, \langle w_s \rangle$$
are pairwise non-conjugate in $F(A)$.
3. The words v_i, w_j are not proper powers in $F(A)$.
4. $|w_j| \geq 30$ for $j = 1, \ldots, s$.
5. The group presentation
$$\langle a_1, \ldots, a_r |\, v_1, \ldots, v_r, w_1, \ldots, w_s \rangle$$
satisfies the $C'(\frac{1}{12})$ small-cancellation condition.

THEOREM 4.3: *Suppose the sequence $v_1, \ldots, v_r, w_1, \ldots, w_s$ satisfies condition D. As in the above definition, let the endomorphism $\phi\colon F(A) \longrightarrow F(A)$ be induced by $\phi(a_i) = v_i$, $i = 1, \ldots, r$. Denote $w_{j,n} = \phi^n(w_j)$ for $n \geq 1$, $j = 1, \ldots, s$. We also denote $w_{j,0} = w_j$ for $j = 1, \ldots, s$.*

Then the following group presentation

$$H = \langle a_1, \ldots, a_r |\, w_{j,n} = 1 \text{ for every } n \geq 0, j = 1, \ldots, s \rangle \tag{1}$$

satisfies the $C(6)$ small-cancellation condition.

For the remainder of this section we assume that the hypotheses of Theorem 4.3 are satisfied. We will first prove a series of auxiliary lemmas and then give the proof of Theorem 4.3 at the end of this section.

Notation 4.4: We will use the notation $a_{-i} = a_i^{-1}$ and $v_{-i} = v_i^{-1}$ for $i = 1, \ldots, r$. We also denote $w_{-j} = w_j^{-1}$ for $j = 1, \ldots, s$.

For every $i = \pm 1, \ldots, \pm r$ and $n \geq 1$ we let $a_{i,n} = \phi^n(a_i)$. We will refer to a cyclic permutations of $w_{j,n}$ as a **relator of grade** n for H (where $n \geq 0$).

Definition 4.5 (Occurrence): Let w, u be freely reduced words in $A^{\pm 1}$. An **occurrence** of u as a subword of w is a presentation $w = w'uw''$. Two occurrences $w = w'uw''$ and $w = z'uz''$ of u in w are said to be the same if $w' = z', w'' = z''$.

Let u be a freely reduced word and let w be a freely and cyclically reduced word. An **occurrence** of u as a subword of the cyclic word $[w]$ is a presentation $w^* = uw'$ where w^* is a cyclic permutation of w. Two occurrences $w^* = uw'$ and $w^{**} = uz'$ of u in $[w]$ are said to be the same if $w' = z'$.

Definition 4.6 (Residue): Let $n \geq 1$ and $W = \phi^n(w)$ for a freely reduced word w in $A^{\pm 1}$. Let u be a subword of w where $w = w'uw''$. We say that the occurrence $W = \phi^n(w')\phi^n(u)\phi^n(w'')$ of $\phi^n(u)$ in W is the **residue** of the occurrence $w = w'uw''$ of u in w.

Also, in this case we will say that the occurrence $\phi^n(u)\phi^n(w''w')$ of $\phi^n(u)$ in $[W] = [\phi^n(u)\phi^n(w''w')]$ is the **residue** of the occurrence $uw''w'$ of u in $[w] = [uw''w']$.

LEMMA 4.7: *Let w be a freely reduced word in the $A^{\pm 1}$. Then the following holds:*

1. *Any occurrence of $v_i^{\pm 1}$ as a subword of $\phi(w)$ is a residue in $\phi(w)$ of an occurrence of $a_i^{\pm 1}$ in w.*
2. *Suppose u is a nontrivial freely reduced word in $A^{\pm 1}$. If $\phi(u)$ is a subword of $\phi(w)$, then this occurrence of $\phi(u)$ in $\phi(w)$ is a residue of an occurrence of u as a subword of w. That is to say, if $\phi(w) = y'\phi(u)y''$ then $w = w'uw''$ where $p = \phi(w')$ and $q = \phi(w'')$.*

Proof: Part 1 follows from the small-cancellation hypotheses D of Theorem 4.3.

Indeed, suppose some v_i is a subword of $\phi(w)$ and assume that part 1 of Lemma 4.7 does not hold in this case. Let $w = x_1 \cdots x_s$ where $x_k \in A^{\pm 1}$ is the freely reduced form of w in $F(A)$. Let $V_k = \phi(x_k)$ for each k. Since ϕ is an immersion, the word $\phi(w) = V_1 \cdots V_s$ is freely reduced in $F(A)$ as well.

The $C'(\frac{1}{12})$ assumption in Condition D implies that the subword v_i of $\phi(w)$ cannot be contained and cannot contain any V_k. Thus $v_i = v'v''$, where v', v'' are nonempty words such that v' is a terminal segment of V_k and v'' is an initial segment of V_{k+1} for some k. One of the words v', v'' has length at least half of $|v_i|$. We will assume that $|v'| \geq (1/2)|v_i|$ and it will be clear that the opposite case is completely analogous.

Thus $|v'| \geq (1/2)|v_i|$ and v' is an initial segment of $v_i = \phi(a_i) = v'v''$ and is a terminal segment of $V_k = \phi(x_k) = uv'$. By the $C'(\frac{1}{12})$ condition v' is not a piece, which means that $v'v'' = v'u$ and $v'' = u$. Thus $\phi(x_k) = v''v'$ and $\phi(a_i) = v'v''$. That is, $\phi(x_k)$ is a cyclic permutation of $\phi(a_i)$.

We have assumed that all the words v_j are not a cyclic permutation of $v_m^{\pm 1}$ for $j \neq m$. Since a word in a free group cannot be a cyclic permutation of its inverse, this means that $x_k = a_i$ and $v'v'' = v''v'$. Since both v', v'' are nontrivial and the word $v'v''$ is freely reduced in $F(A)$, the equality $[v', v''] = 1$ means that v', v'' are powers of the same element in $F(A)$, and so $v_i = \phi(a_i)$ is a proper power, contrary to our assumptions in Condition D. Thus Part 1 of Lemma 4.7 has been proved.

It is now easy to see that Part 1 implies Part 2. ∎

LEMMA 4.8: *Let $n \geq 1$. Suppose that for some freely reduced words u, w in $A^{\pm 1}$, $\phi^n(u)$ is a subword of $\phi^n(w)$. Then this subword is a residue of an occurrence of u as a subword of w. To be precise, if $\phi^n(w) = W'\phi^n(u)W''$ then $w = w'uw''$ with $\phi^n(w') = W'$ and $\phi^n(w'') = W''$.*

Proof: We will prove this statement by induction on n. For $n = 1$ it immediately follows from Lemma 4.7.

Let $n > 1$ and suppose that the statement has been proved for all smaller values of n. Thus $\phi(\phi^{n-1}(u))$ is a subword of $\phi(\phi^{n-1}(w))$, that is $\phi(\phi^{n-1}(w)) = W'\phi(\phi^{n-1}(w))W''$. Denote $\alpha = \phi^{n-1}(w)$ and $\beta = \phi^{n-1}(w)$. By induction $\alpha = \alpha'\beta\alpha''$, where $\phi(\alpha') = W'$ and $\phi(\alpha'') = W''$.

Since $\beta = \phi^{n-1}(v)$ is a subword of $\alpha = \phi^{n-1}(w)$, the inductive hypothesis also implies that $w = w'uw''$ with $\phi^{n-1}(w') = \alpha'$ and $\phi^{n-1}(w'') = \alpha''$. Together with $\phi(\alpha') = W'$ and $\phi(\alpha'') = W''$ this implies that $\phi^n(w') = W'$ and $\phi^n(w'') = W''$ as required. ∎

LEMMA 4.9: *Suppose that for some $n \geq 1$ and $i = \pm 1, \ldots, \pm r$ the word $a_{i,n}$ is a subword of the cyclic word $[w_{j,n}]$. Then this occurrence of $a_{i,n}$ is the residue in $w_{j,n}$ of some occurrence of a_i in $[w_j]$.*

Proof: This is a special case of Lemma 4.8, since if a_{i_n} is a subword of the cyclic word $[w_{j,n}]$ then a_{i_n} is a subword of the word $w_{j,n}w_{j,n} = \phi^n(w_j^2)$. ∎

LEMMA 4.10: *Suppose that for a nontrivial freely reduced word u in $A^{\pm 1}$ the word $\phi^n(u)$ is a subword of a cyclic word $[w_{j,n}]$, where $n \geq 1$. Then this occurrence of $\phi^n(u)$ is the residue of an occurrence of u as a subword of a cyclic word $[w_j]$.*

Proof: This follows immediately from Lemma 4.9. ∎

LEMMA 4.11: *Let u be a nontrivial freely reduced word in $A^{\pm 1}$. Then for $m > n \geq 1$, the word $\phi^m(u)$ is not a subword of $[w_{j,n}]$.*

Proof: Suppose $\phi^m(u) = \phi^n(\phi^{m-n}(u))$ is a subword of the cyclic word $[w_{j,n}]$. By Lemma 4.10, $\phi^{m-n}(u)$ is a subword of $[w_j]$. Since $m - n > 0$ and u is nontrivial, this means that some v_i is a subword of $[w_j]$. This obviously contradicts the small-cancellation hypothesis in Theorem 4.3. ∎

LEMMA 4.12 (Overlap with identical grade): *Let $n \geq 1$ and let w be a subword of both $[w_{j,n}]$ and $[w_{k,n}]$, and assume that if $j = k$ then the occurrences of w are distinct. Suppose u is a subword of $[w_j]$ such that its residue $\phi^n(u)$ is contained in w.*

Then $|u| < \frac{1}{12}|w_j|$.

Proof: Lemma 4.10 implies that the word $\phi^n(u)$, considered as a subword $[w_{k,n}]$, corresponds to the image of a subword u of $[w_k]$. Thus u is a subword of both $[w_j]$ and $[w_k]$. If $j \neq k$, then by the small-cancellation hypothesis of Theorem 4.3 we have $|u| < (1/12)|w_j|$ as required. A similar argument holds if $k = j$. ∎

LEMMA 4.13: *Let $m > n \geq 1$. Suppose that for some nontrivial freely reduced u the word $\phi^n(u)$ occurs as a subword of a cyclic word $[w_{j,m}]$. Then u is a subword of the cyclic word $[\phi^{m-n}(w_j)]$.*

Proof: Since $w_{j,m} = \phi^n(\phi^{m-n}(w_j))$, this Lemma is an easy corollary of Lemma 4.8. ∎

LEMMA 4.14 (Overlap with higher grade): *Let $m > n \geq 1$. Let p be a word which occurs as a subword of both the cyclic words $[w_{j,n}]$ and $[w_{k,m}]$.*

Then for any subword u of $[w_j]$ such that $\phi^n(u)$ is a subword of p, we have $|u| < (1/6)|w_j|$.

Proof: Suppose that u is a subword of $[w_j]$ such that $\phi^n(u)$ is contained in p. Then by Lemma 4.13 the word u is a subword of the cyclic word $[\phi^{m-n}(w_k)]$.

Hence u is a subword of a word in $v_1, \ldots, v_r$. No v_i can be a subword of $[w_j]$, thus of u, by the small-cancellation assumption D.

Thus u is a subword of the concatenation $v_q v_s$, $q \neq -s$. Therefore at least half of u is an initial (or terminal) segment of some v_i. Consequently, at least half of u is a subword of $[w_j]$ and $[v_i]$. Hence by the small-cancellation hypothesis $|u|/2 < (1/12)|w_j|$ and so $|u| < (1/6)|w_j|$, as required. ∎

LEMMA 4.15 (Pieces in grade > 1 relators): *Let $n \geq 1$. Suppose w is a subword of a cyclic word $[w_{j,n}]$ and that w is a piece with respect to presentation (1). Then if u is a subword of a cyclic word $[w_j]$ and $\phi^n(u)$ is a subword of w, then $|u| < (1/6)|w_j|$.*

Proof: The subword w of the cyclic word $[w_{j,n}]$ can arise as a piece in four ways.

First, suppose that w is a piece because it occurs in another way as a subword of a grade n relator. Then by Lemma 4.12, $|u| < (1/6)|w_j|$.

Second, suppose that w is a piece because it occurs as a subword of a grade m relator where $m > n$. Then by Lemma 4.14, $|u| < (1/6)|w_j|$.

Third, suppose that w is a piece because it occurs as a subword of a grade m relator where $1 \leq m < n$. Then by Lemma 4.12, u must be the empty subword.

Fourth, suppose that w is a piece because it occurs as a subword of a cyclic word $[w_k]$. By the small-cancellation hypothesis of Theorem 4.3 a word v_i cannot be a subword of a cyclic word $[w_k]$. It follows that if u is nontrivial then w cannot contain $\phi^n(u)$ because $\phi^n(u)$ is a nontrivial freely reduced word in $v_1, \ldots, v_r$. Therefore u is the empty word.

Thus in all four cases Lemma 4.15 holds. ∎

LEMMA 4.16 (Piece in grade 0 relator): *Let p be a subword of a cyclic word $[w_j]$ such that p is a piece with respect to presentation (1). Then $|p| < (1/6)|w_j|$.*

Proof: If p is a piece because it is a subword of another grade zero relator, then $|p| < (1/12)|w_j|$ by the small-cancellation hypothesis of Theorem 4.3.

Suppose now that p is a piece because it occurs as a subword of $[w_{k,n}]$ for $n > 1$ and $k = \pm 1, \ldots, \pm s$. Since p is a subword of $[w_j]$, the small-cancellation condition D implies that v_i cannot occur as a subword of p. But p is a subword of $[w_{k,n}]$, which is a word in $v_{\pm 1}, \ldots, v_{\pm r}$ (since $n \geq 1$). Hence p is a subword of the concatenation $v_s v_q$ for $s \neq -q$. Therefore at least half of p is an initial or terminal segment of one of v_i. So at least a half of p is a subword of both $[w_j]$ and $[v_i]$. Hence, by the small-cancellation hypothesis of Theorem 4.3, $(1/2)|p| < (1/12)|w_j|$ and so $|p| < (1/6)|w_j|$, as required. ∎

The following proposition completes the proof of Theorem 4.3.

PROPOSITION 4.17: *Presentation (1) satisfies the $C(6)$ small-cancellation condition.*

Proof: We first consider the case where w is a cyclic permutation of $w_{j,n}$ for $n \geq 1$, $j = \pm 1, \ldots, \pm s$. Suppose that w is a product $p_1 \cdots p_d$ of fewer than six pieces, so that $d \leq 5$. For $1 \leq i \leq d$, there is a largest subword u_i of $[w_j]$ such that $\phi^n(u_i)$ is a subword of p_i. It is easy to see that for $1 \leq i < d$, there is at most one letter in w_j between u_i and u_{i+1}, and similarly at most one letter between u_d and u_0 in the cyclic word $[w_j]$. It follows that $|w_j| \leq \sum_{1 \leq i \leq d}(|u_i| + 1)$. Note that, by Lemma 4.15, $|u_i| < \frac{1}{6}|w_j|$ for each i and that $d \leq 5$ by our assumption.

This implies that $|w_j| < \frac{5}{6}|w_j| + 5$, $6|w_j| < 5|w_j| + 30$ and hence $|w_j| < 30$, which contradicts the final hypothesis of Theorem 4.3 that $|w_j| \geq 30$.

Suppose now that w is a cyclic permutation of w_j and that w is a product of fewer than six pieces (with respect to presentation (1)). By Lemma 4.16 each of these pieces has length less than $(1/6)|w_j|$. Therefore their total length is less than $(5/6)|w_j| < |w_j|$, which is impossible. ∎

5. Failure of co-Hopficity

THEOREM 5.1: *Let $s \geq 1$ and let $w_1, \ldots, w_s$ be freely and cyclically reduced, root-free words in $F(A)$ such that $[w_k] \neq [w_j^{\pm 1}]$ for $k \neq j$. Let ϕ: $F(A) \longrightarrow F(A)$, $\phi(a_i) = v_i$ for $i = 1, \ldots, r$, be an immersion.*

Let G be a group given by the presentation

$$(2) \qquad G = \langle a_1, \ldots, a_r, t \mid w_j, a_i^t = v_i : 1 \leq i \leq r, 1 \leq j \leq s \rangle.$$

Suppose that this presentation of G satisfies the $C'(1/26)$ small cancellation condition. Suppose also that $|w_j| > 30$ for $j = 1, \ldots, s$ and $|v_i| > 36$ for $i = 1, \ldots, r$ and that each letter a_i occurs in at least one of w_j.

Let $H = \langle a_1, \ldots, a_r \rangle \subset G$.

Then $H^t = \langle v_1, \ldots, v_r \rangle \subsetneq H$. In particular H is not co-Hopfian.

Proof: Note first that the sequence $v_1, \ldots, v_r, w_1, \ldots, w_s$ satisfies Condition D. Indeed, v_i cannot be a proper power and v_i is not conjugate to $v_j^{\pm 1}$ (for $i \neq j$) because the presentation of G satisfies the $C'(1/26)$ condition. For the same reason v_i is not a cyclic permutation of $w_j^{\pm 1}$. Also, by assumption w_j is not a cyclic permutation of $w_k^{\pm 1}$ when $j \neq k$ and the elements w_j are not proper powers.

Hence the infinite cyclic subgroups $\langle v_1\rangle, \ldots, \langle v_r\rangle, \langle w_1\rangle, \ldots, \langle w_s\rangle$ are pairwise non-conjugate in $F(A)$.

It is easy to see that the set $P = \{[v_1^{\pm 1}], \ldots, [v_r^{\pm 1}], [w_1^{\pm 1}], \ldots, [w_s^{\pm 1}]\}$ satisfies the $C'(1/12)$ small-cancellation condition.

Indeed, if u is a piece with respect to P then at least half of u is a piece with respect to presentation (2). If u is a piece with respect to P by virtue of being a subword of $[v_i^{\pm 1}]$, then

$$(1/2)|u| < (1/26)(|t^{-1}a_i t v_i^{-1}|) = (1/26)(|v_i| + 3),$$

which implies $|u| < (1/12)|v_i|$ since $|v_i| > 36$. If u is a piece with respect to P because u is a subword of $[w_j^{\pm 1}]$, then $(1/2)|u| < (1/26)|w_j|$ and hence $|u| < (1/13)|w_j| < (1/12)|w_j|$. Thus P indeed satisfies the $C'(1/12)$ condition and we have verified Condition D.

Therefore both Theorem 3.1 and Theorem 4.3 apply. That is, H has presentation (1) and this presentation satisfies condition $C(6)$.

We will show that $H^t \neq H$ by proving that $a_1 \notin H^t$. Arguing by contradiction, suppose that $a_1 \in H^t$. Then for some freely reduced word f in the $a_i^{\pm}$, we have $a_1 = \phi(f)$ in G (and so in H). Among all such words we choose f such that the relation $a_1^{-1}\phi(f) = 1$ has the smallest area with respect to presentation (1) of H.

Consider a disc-diagram D over presentation (1) of H for the relation $a_1^{-1}\phi(f) = 1$.

First observe that D has at least one 2-cell. Indeed, if D is a tree, then a_1 is freely equivalent to $\phi(f)$. But ϕ is an immersion, and so $\phi(f)$ is already freely reduced, and consequently $|\phi(f)| > 36$ since by hypothesis $|\phi(a_i)| = |v_i| \geq 36$. Thus it is impossible for $\phi(f)$ and a_1 to be the same word.

Because ϕ is an immersion, the word $\phi(f)$ corresponds to an immersed path in the standard 2-complex X_H of the presentation for H. It follows that the disc-diagram for $a_1^{-1}\phi(f) = 1$ can have at most one spur, and this spur occurs at one endpoint of the obvious a_1 edge in ∂D.

To be more precise, let $\phi(f) = v_{i_1} \cdots v_{i_t}$. Then the word

$$a_1^{-1}\phi(f) = a_1^{-1} v_{i_1} \cdots v_{i_t}$$

is close to being cyclically reduced, since the small-cancellation hypothesis on v_i implies that the free cancellation in the product $v_{i_t} a_1^{-1} v_{i_1}$ involves at most one third of v_{i_1} and v_{i_t}. Since ϕ is an immersion, the word $v_{i_1} \cdots v_{i_t}$ is already freely reduced. Thus D indeed can have at most one spur which ends at one of the endpoints of the a_1 edge in the boundary of D and which is the last edge in an arc of edges which is of length at most $(1/3)\min\{|v_{i_1}, v_{i_t}|\}$.

We let D' be the largest subdiagram of D which contains no spurs. In particular D can be obtained from D' by adding a (possibly trivial) arc of edges at some point p in $\partial D'$. (If D already has no spurs we choose p to be the vertex in ∂D corresponding to the start of the path $a_1^{-1}\phi(f)$.)

We now apply Greendlinger's Lemma to the diagram D' to obtain a pair of 2-cells C_1 and C_2 with the property that for each i, ∂C_i intersects $\partial D'$ in a subpath Q_i of the boundary cycle of D'. Furthermore, for each i, the complement of Q_i in ∂C_i is a path S_i which is the concatenation of at most three pieces in D'. In case $D = D'$ then the a_1 edge can be contained in the interior of at most one of Q_1 and Q_2. Similarly, if $D \neq D'$ then the point p cannot be contained in the interior of both Q_1 and Q_2. Therefore, we can assume that neither the a_1 edge nor the point p (in case D has a spur) is contained in the interior of the Q_1 path. It follows that Q_1 is a subpath of the boundary cycle of D and Q_1 does not contain the a_1 edge in its interior.

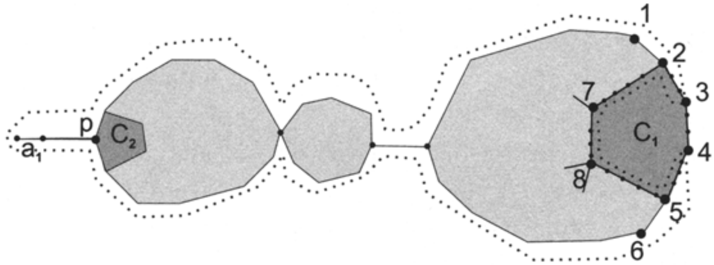

Figure 3. Illustrated above is a disc-diagram D for the proof of Theorem 5.1 in the case where D has a spur. The subdiagram $\bar{D}$ is the part of the disc-diagram to the right of the 0-cell labeled 'p'. The path Q_1 travels along the 0-cells labeled $2,3,4,5$. The path S_1 travels along the 0-cell labeled $2,7,8,5$. The specific v_i path that is of interest in the proof is the subpath of Q_1 whose endpoints are $3,4$. The cell C_1 retracts onto the part of ∂C_1 which is the path traveling along $3,2,7,8,5,4$. The new diagram $\bar{D}$ is obtained from D by removing the open 2-cell C_1 and removing the interior of the v_i path. The boundary path for $\bar{D}$ is the path $a_1^{-1}\phi(\bar{f})$ which is indicated by the dotted path. It is identical to $a_1^{-1}\phi(\bar{f})$ except near C_1. The difference is that the old boundary path travels along the 0-cells $1,2,3,4,5,6$, but the new boundary path travels along $1,2,3,2,7,8,5,4,5,6$.

Now the path S_1 is the concatenation of at most three pieces in D and hence in X_H. By the $C(6)$ condition, the path Q_1 is not the concatenation of fewer than three pieces of X_H. It follows that the path Q_1 must contain at least one complete v_i subpath. Indeed, if Q_1 contained only parts of v_i words, then it could consist of the concatenation of at most two such parts, but since each such part is a piece, it would be the concatenation of at most two pieces, which is impossible.

We note that C_1 must correspond to a relator $w_{j,n}$ where $n \geq 1$. Indeed, v_i cannot be a subword of the cyclic word $w_{j,n}$ because then v_i would be a long piece of $a_i^t = v_i$ which would violate the $C'(\frac{1}{26})$ condition on the presentation for G. Consequently we can assume that C_1 corresponds to a relator $w_{j,n}$ where $n \geq 1$.

Since $n \geq 1$, the boundary cycle of C_1 corresponds to a certain relation $v_i = \phi(g)$ where g is a certain subword of $w_{j,(n-1)}$, and we are careful to choose the v_i corresponding to the v_i path in ∂D that we have specified above. In a corresponding fashion, the relator $w_{j,(n-1)}$ corresponds to a relation $a_i = g$.

By pushing the v_i path inwards, towards the $\phi(g)$ path in ∂C_1, we obtain a retraction of C_1 onto $\partial C_1 - v_i$, and this induces a retraction of D onto a new disc-diagram $\bar{D}$ with smaller area.

Furthermore, since we have only changed the boundary cycle of D by substituting $\phi(g)$ for $\phi(a_i)$, we see that $\bar{D}$ is a disc-diagram for the relation $a_1 = \phi(\bar{f})$ where $\bar{f}$ is obtained from f by substituting g for the occurrence of the a_i in f that corresponds to the v_i path in ∂D.

This provides the desired contradiction of the minimality of area of D. Specifically, if we freely reduce $\bar{f}$ to obtain a word y, then a disc-diagram for the relation $a_1 = \phi(y)$ can be obtained from $\bar{D}$ by pushing the spurs of $\bar{D}$ inwards. ∎

LEMMA 5.2: *Let G be a two-generated non-elementary subgroup of a torsion-free word-hyperbolic group.*

(a) *If the abelianization of G is finite then G has no nontrivial essential cyclic splittings.*
(b) *If the abelianization of G is trivial then G has no nontrivial cyclic splittings.*

Proof: (a) Suppose first G has finite abelianization. Since G is two-generated and torsion-free, this implies that G is freely indecomposable. Observe that if H splits as an HNN extension, then the obvious retraction map onto the cyclic subgroup generated by the stable letter gives a surjection $H \to \mathbb{Z}$, which is impossible because the abelianization of H is finite.

Suppose that H admits a nontrivial essential cyclic splitting. Since H is not an HNN-extension, this implies that $H = A *_C B$ where $C = \langle c \rangle$ is an infinite cyclic group which has infinite index in both A and B.

Assume first that C is a maximal cyclic subgroup of both A and B. Since A and B are subgroups of a torsion-free word-hyperbolic group, this implies that C is malnormal in both A and B [KM98]. Thus $G = A *_C B$, where C is a malnormal and proper subgroup of both factors. Therefore by a theorem of Karrass and Solitar [KS71] G cannot be generated by two elements, contrary to our assumption.

Assume now that C is maximal in one of the factors, but not in the other. Say $c = a^n = b$, where $n > 1$ and where $\langle a \rangle, \langle b \rangle$ are maximal cyclic subgroups of A and B respectively. Again, by maximality, $\langle a \rangle$ is malnormal in A and $\langle b \rangle$ is malnormal in B.

Then

$$G = A *_{a^n=b} B = A *_{a=z} (\langle z \rangle *_{z^n=b} B) = A *_{a=z} B'.$$

Note that now the infinite cyclic subgroup $\langle a \rangle$ is maximal in both A and B', which is impossible by the previous case.

Assume now that C is not maximal cyclic in either of A, B. Then $c = a^n = b^m$ where $n > 1, m > 1$ and where $\langle a \rangle, \langle b \rangle$ are maximal cyclic subgroups of A and B respectively. Thus $G = A *_{a^n=b^m} B$. In this case it is easy to see that $\langle a^n, ab \rangle \cong \mathbb{Z} \times \mathbb{Z}$, which is impossible since G is a subgroup of a word-hyperbolic group.

Thus G does not admit an essential cyclic splitting.

(b) Suppose now that G has trivial abelianization.

By (a) we already know that G has no essential cyclic splittings. Suppose, however, that G has a nontrivial cyclic splitting.

We know by the argument in the proof of (a) that G cannot split as an HNN-extension.

Then G can be represented as a nontrivial amalgamated free product $G = A *_C B$ where C is infinite cyclic and has finite index (greater than one) in at least one of the factors, say B. Thus G has the form $G = A *_{a=c^n} \langle c \rangle$, where $\langle c \rangle$ is an infinite cyclic group and $n > 1$. Hence the quotient of G by the normal closure of A is a finite cyclic subgroup of order n. This contradicts our assumption that G has trivial abelianization. ∎

We are now ready to prove the main result of this paper (cf. Theorem 1.2):

THEOREM 5.3: *There exist a torsion-free $C'(1/26)$ (and so word-hyperbolic) group G and a subgroup H of G such that the following holds.*

(i) *The group H is two-generated and non-elementary.*
(ii) *The group H does not admit a nontrivial cyclic splitting (and so in particular H is freely indecomposable and one-ended).*
(iii) *The group H is conjugate in G to a proper subgroup of itself. Thus H is not co-Hopfian.*

Proof: We apply Theorem 5.1 to the presentation

$$G = \left\langle a_1, a_2, t \,\mathrm{St}\; \begin{matrix} a = [a,b][a^2,b^2]\cdots[a^{100},b^{100}], & a^t = abab^2\cdots ab^{100}a, \\ b = [b,a][b^2,a^2]\cdots[b^{100},a^{100}], & b^t = baba^2\cdots ba^{100}b \end{matrix} \right\rangle.$$

We leave it to the reader to verify that the presentation above satisfies the hypotheses of Theorem 5.1 and therefore $H = \langle a, b\rangle \neq H^t \subset H$ and so H is not co-Hopfian. Note that H is not elementary because, by Remark 3.3, H is not finitely presentable.

The above presentation of G obviously implies that H is perfect, that is, H has trivial abelianization. Therefore by Lemma 5.2, H has no nontrivial cyclic splittings. ∎

6. An alternative construction

The key point in the proof of Theorem 5.1 was to show that the two-generated subgroup H of G had trivial abelianization. This guaranteed that H did not split over $\mathbb{Z}$.

However, if one only needs for G to have no essential splittings over $\mathbb{Z}$, then a more direct argument is available. Namely, we can construct a torsion-free small-cancellation group G and a non-elementary subgroup H of G so that $H^t \leq H$ for some $t \in G$, the abelianization of H is finite and there is a certain homomorphic image of G such that the images of H and H^t are easily seen to be distinct. This shows that $H \neq H^t$ and so H is not co-Hopfian.

We will give a sketch of this argument below.

The following statement is a corollary of the result of S. Adian (see the proof of Theorem 3.7, section VI, [Adi79]).

LEMMA 6.1: *Let $B(2,667) = \langle a, b |\ f^{667} = 1, f \in F(a,b)\rangle$ be the free Burnside group of rank two and exponent 667. Then $B(2,667)$ is infinite and not co-Hopfian. In particular, for the freely reduced words $\alpha = bab^{-1}, \beta = b^2ab^{-2}$ the* *subgroup of $B(2,667)$ generated by α, β is proper in $B(2,667)$ and isomorphic to $B(2,667)$ under the map*

$$\psi\colon B(2,667) \longrightarrow B(2,667),$$
$$\psi(a) = bab^{-1}, \quad \psi(b) = b^2ab^{-2}.$$

We will fix the words $\alpha = aba^{-1}, \beta = a^2ba^{-2}$ provided by Lemma 6.1 till the end of this section.

LEMMA 6.2: *Let G be the group given by the presentation*

$$G = \langle a, b, t \,|\, t^{-1}at = W_1,\ t^{-1}bx = W_2,\ U_1 = 1,\ U_2 = 1\rangle \tag{3}$$

where the words W_1, W_2, U_1, U_2 are chosen so that:

1. *The words W_1 and W_2 are of the form $W_1 = aba^{-1}V_1 \cdots V_m$ and $W_2 = a^2ba^{-2}V_{m+1} \cdots V_n$ where all V_i are 667-th powers of some words in $F(a,b)$.*
2. *The words U_1 and U_2 are products of 667-th powers of some words in $F(a,b)$, but U_1 and U_2 are not themselves proper powers in $F(a,b)$, and the group given by the presentation*

   $$\langle a, b \,|\, [a,b] = 1, U_1 = 1, U_2 = 1\rangle$$

   is finite.
3. *Presentation (3) satisfies the $C'(1/6)$ small-cancellation condition, so that the group G is torsion-free word-hyperbolic.*

Let H be the subgroup of G generated by a, b and let L be the subgroup of G generated by W_1, W_2, that is $L = H^t$.

Then H is one-ended and does not split essentially over, $\mathbb{Z}$. Moreover, $L = H^t \lneqq H$ and so H is not co-Hopfian.

Proof: Note that H is not infinite cyclic. Indeed, if it is an infinite cyclic group then $[a^t, b^t] = 1$ implies $[a,b] = 1$. However, the word $[a,b] = a^{-1}b^{-1}ab$ has length four and it cannot contain more than a half of a defining relator of G, giving a contradiction.

Thus H is not infinite cyclic and so H is non-elementary. Note also that by the choice of U_1, U_2 the group H has finite abelianization.

This implies that H is one-ended. Indeed, if H has infinitely many ends, then H splits as a nontrivial free product $H = A * B$ (since H is torsion-free). Since H is two-generated, Grushko's theorem implies that both A and B are one-generated, that is infinite cyclic. Then H is a free group of rank two, which contradicts the fact that the abelianization of H is finite.

Since H is a two-generated, freely indecomposable subgroup of a torsion-free word-hyperbolic group and H has finite abelianization, Lemma 5.2 implies that H admits no nontrivial essential cyclic splittings.

It remains to establish that $H \neq H^t$.

Let $\bar{G}$ be the quotient of G obtained by factoring the normal subgroup of G generated by all words f^{667} where $f \in F(a,b)$. Let ρ: $G \longrightarrow \bar{G}$ be the associated canonical epimorphism. For $g \in G$ we will denote $\rho(g)$ by $\bar{g}$. Also denote $\bar{H} = \rho(H), \bar{L} = \rho(L)$. Then the group $\bar{G}$ has the following presentation:

$$\bar{G} = \langle \bar{a}, \bar{b}, \bar{t} \,|\, (\bar{t})^{-1}\bar{a}\bar{t} = \bar{\alpha}, (\bar{t})^{-1}\bar{b}\bar{t} = \bar{\beta}, \bar{f}^{667} = 1 : \forall \bar{f} \in F(\bar{a}, \bar{b}) \rangle. \tag{4}$$

Moreover, $\bar{H} = \langle \bar{a}, \bar{b} \rangle$ and $\bar{L} = \langle \bar{\alpha}, \bar{\beta} \rangle$. By the choice of α, β in Lemma 6.1 both groups $\langle \bar{a}, \bar{b} \rangle$ and $\langle \bar{\alpha}, \bar{\beta} \rangle$ are isomorphic to the free Burnside group $B(2, 667)$, $\langle \bar{\alpha}, \bar{\beta} \rangle \lneqq \langle \bar{a}, \bar{b} \rangle$ and presentation (4) is a strictly ascending HNN-extension of $B(2, 667)$. In particular, we have $\bar{L} \lneqq \bar{H}$ and hence $H^t = L \lneqq H$. Thus H is not co-Hopfian and Theorem 6.2 is proved. ∎

Remark 6.3: The proof of our main result, Theorem 1.2, could probably have been significantly simplified if there existed a *two-generated* finitely presentable group Q which is not co-Hopfian and which has trivial abelianization. (A variation of Rips' construction could possibly be used then.) At the moment, however, we do not know of any such examples and they seem to be difficult to construct. Even if one drops the requirement of finite presentability, then the two-generated subgroup H provided in the proof of Theorem 1.2 appears to be the first such example. Moreover, even if one further relaxes the restriction on such a group, allowing the abelianization to be finite rather than trivial, there are few available examples. Basically they all come from the difficult result of S. Adian [Adi79], mentioned above, which states that many free Burnside groups are not co-Hopfian. For example, the group $B(2, 667)$ is not co-Hopfian and it has finite but nontrivial abelianization. ∎

References

[Adi79] S. I. Adian, *The Burnside Problem and Identities in Groups,* Springer-Verlag, Berlin, 1979. (Translated from the Russian by J. Lennox and J. Wiegold.)

[Aea91] J. M. Alonso et al., *Notes on word hyperbolic groups,* in *Group Theory from a Geometrical Viewpoint (Trieste, 1990)* (H. Short, ed.), World Sci. Publ., River Edge, NJ, 1991, pp. 3–63.

[Bau66] B. Baumslag, *Intersections of finitely generated subgroups in free products*, Journal of the London Mathematical Society **41** (1966), 673–679.

[BF95] M. Bestvina and M. Feighn, *Stable actions of groups on real trees*, Inventiones Mathematicae **121** (1995), 287–321.

[Bra] N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, Journal of the London Mathematical Society **60** (1999), 461–480.

[DP98] T. Delzant and L. Potyagailo, *Small splittings of Kleinian groups and embedding properties*, preprint, 1998.

[Ger96] S. M. Gersten, *Subgroups of word hyperbolic groups in dimension* 2, Journal of the London Mathematical Society (2) **54** (1996), 261–283.

[Gre60] M. Greendlinger, *Dehn's algorithm for the word problem*, Communications on Pure and Applied Mathematics **13** (1960), 67–83.

[KM98] O. Kharlampovich and A. Myasnikov, *Hyperbolic groups and free constructions*, Transactions of the American Mathematical Society **350** (1998), 571–613.

[KS71] A. Karrass and D. Solitar, *The free product of two groups with a malnormal amalgamated subgroup*, Canadian Journal of Mathematics **23** (1971), 933–959.

[LS77] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin, 1977.

[MW] J. P. McCammond and D. T. Wise, *Coherence, local quasiconvexity and the perimeter of 2-complexes*, preprint.

[OP98] Ken'ichi Ohshika and Leonid Potyagailo, *Self-embeddings of Kleinian groups*, Annales Scientifiques de l'École Normale Supérieure (4) **31** (1998), 329–343.

[Pot97] L. D. Potyagaĭlo, *The co-Hopf property and the embedding of Kleinian groups into themselves*, in *Algebra, Geometry, Analysis and Mathematical Physics (Novosibirsk, 1996)*, Izdat. Ross. Akad. Nauk Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1997, p. 180 (Russian).

[Rip82] E. Rips, *Subgroups of small cancellation groups*, The Bulletin of the London Mathematical Society **14** (1982), 45–47.

[RS94] E. Rips and Z. Sela, *Structure and rigidity in hyperbolic groups. I*, Geometric and Functional Analysis **4** (1994), 337–371.

[Sel97] Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank* 1 *Lie groups. II*, Geometric and Functional Analysis **7** (1997), 561–593.

[Sel99] Z. Sela, *Endomorphisms of hyperbolic groups. I. The Hopf property*, Topology **38** (1999), 301–321.

[Wis98] D. T. Wise, *Incoherent negatively curved groups*, Proceedings of the American Mathematical Society **126** (1998), 957–964.